

# A note on the Full Counting Statistics of paired fermions

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We study the trace of the exponentials of general fermion bi-linears, including pairing terms, and including non Hermitian forms. In particular, we give elementary derivations for determinant and pfaffian formulae for such traces, and use these to obtain general expressions for the full counting statistics in states associated with quadratic Hamiltonians including BCS-like pairing terms and fermion parity in a prescribed region or set of modes. We also derive pfaffian expressions for state overlaps and counting statistics in states built out of the vacuum by creation of pairs of particles.

## I. INTRODUCTION.

Fluctuations of observables in quantum mechanical systems carry important information about a variety of properties, such as the nature of transport mechanisms in mesoscopic systems or spin excitations in magnetic systems, and supply a tool for studying physical properties both in equilibrium and non-equilibrium settings. An important example for such is the use of shot noise, i.e. non-equilibrium current fluctuations, to get information about the nature of charge carriers in complicated interacting situations such as fractional quantum Hall systems, in which collective excitations carry a fractional charge. Additional, more detailed information about condensed matter systems, may be hidden in higher order noise correlations. This information becomes accessible as our ability to carry out precision measurements increases. The full distribution functions of transmitted charges, known as the "Full Counting Statistics" (FCS) have been of interest in the description of transport in mesoscopic physics, and raise interesting problems both about the nature of transport and the nature of measurements themselves.

The theory of FCS was pioneered in the celebrated work of Levitov and Lesovik [1], where a beautiful formula for the FCS of transport through a junction has been derived. Subsequently, FCS has seen intense work in mesoscopic physics [1–7] and in cold atom systems [8, 9]. The formulation of the FCS results for free fermions presented in [10], has

been successfully used to simplify FCS and related calculations and has proved useful also in other contexts, describing time dependent problems involving non-interacting fermions. For example, the FCS formalism has been found to play a useful role in the Fermi-edge problem [11, 12], and has been useful as a numerical tool for other time dependent problems, for example in studies of thermalization and decoherence of metallic leads [13], as a tool to characterize correlations and phases [14], identify dynamical phases in the resonant-level model [15] as well as characterizing non-equilibrium such as the evolution of systems following a quench [16]. FCS was studied theoretically and measured experimentally in several mesoscopic systems, such as tunnel junctions and quantum point contacts [17–19] where non-gaussian fluctuations [18] and counting statistics of single electrons [20] were measured. Some attention has also been devoted to states where pairing terms appear in the Hamiltonian, to treat situations where superconductivity is present. In such situations transitions between electron holes and Cooper pairs through Andreev tunneling is present (see, e.g. [21–24]). Recent experimental progress has been reported in the measurements of FCS of Andreev tunneling in [25]. Shot noise signatures for systems with fractionalized charges have been proposed theoretically [26–28] and used to experimentally access fractional charges [29–31]. On the mathematical side, much work has been done to understand the thermodynamic limit of FCS on a more rigorous level, see, e.g [32–36].

In a broader context, it has also been demonstrated that fluctuations, and the FCS of charge or other conserved quantities (such as, for example, block magnetization in certain spin chains), may contain information about the full entanglement scaling of a system split into two parts. It was shown how one can compute entanglement entropy and Renyi entropies from FCS for certain systems in [37], a measurement of entanglement entropy in a transport experiment using FCS was suggested in [38] and other systems were discussed in [39, 40].

Many of the above advances hinge on the efficient calculation of partition function like objects - traces of exponents of quadratic fermion Hamiltonians. However, paired Hamiltonians possess an additional layer of complexity compared to the non-interacting fermions, due to the lack of charge conservation. Algebraically, the lifting of single particle dynamics is non-unique, (the group of Bogolubov transformations is not simply connected) resulting in sign ambiguities, in some simple formulas as presented below.

Here we present a rather simple closed formula, Eq (3), for the trace of a single exponent of quadratic fermion operators, where the sign ambiguity is not present. For products of

exponents and most practical purposes, in the equations presented below, (25), the sign can be determined by simple analyticity arguments. In addition we supply simple formulas for full counting statistics and overlaps in pairing states as described in Eq. (58) and Eq. (59).

## II. THE TRACE OF THE EXPONENTIAL OF A FERMION BILINEAR

In working with Hamiltonians which include pairing terms it is convenient to represent the fermions using Majorana operators. We consider a Hilbert space  $\mathfrak{h}$  of possible single particle states (modes), and associated fermionic operators  $a_i^\dagger$ . For an  $n$ -dimensional  $\mathfrak{h}$ , we have label a basis of states by  $i = 1, \dots, n$ . Define  $2n$  Majorana operators by

$$\begin{aligned} c_i &= a_i^\dagger + a_i \quad i = 1, \dots, n, \\ c_i &= i(a_i^\dagger - a_i) \quad i = n + 1, \dots, 2n, \end{aligned} \quad (1)$$

with the  $c_i$  obeying the Clifford relations:

$$c_i c_j + c_j c_i = 2\delta_{ij}. \quad (2)$$

To define taking traces below, we use the algebraic relations, and take  $\text{Tr} I = 2^n$  (alternatively, one may represent the algebra on a  $2^n$  dimensional Hilbert space explicitly in terms of Pauli matrices, augmented by Jordan-Wigner strings).

In this section we show that for a  $2n$  dimensional, antisymmetric matrix  $A \in \text{Skew}(2n, \mathbb{C})$ , the following formula holds:

$$\mathcal{Z}(A) \equiv \text{Tr} e^{A_{ij} c_i c_j} = \frac{\text{Pf}(e^{-4A} - e^{4A})}{\text{Pf}(e^{-2A} - e^{2A})}. \quad (3)$$

where, Pf is the Pfaffian defined by

$$\text{Pf}(M) = \frac{1}{2^n n!} \sum_{\mathcal{P} \in \mathcal{S}_{2n}} (-1)^{\mathcal{P}} M_{P_1 P_2} M_{P_2 P_3} \dots M_{P_{2n-1} P_{2n}} \quad (4)$$

where  $\mathcal{S}_{2n}$  is the permutation group on  $2n$  elements.

In addition, we note a simpler formula, which is correct up to a sign ambiguity:

$$\mathcal{Z}(A) = \sqrt{\det(1 + e^{4A})} \quad (5)$$

Note the appearance of the square root in the determinant equation. In this equation, the sign of the determinant has to be determined. In many practical calculations the sign can

be determined as follows. Consider  $\mathcal{Z}(\lambda A)$ : it follows from its definition as a trace in eq. (3) that  $\mathcal{Z}(\lambda A)$  is an analytic function of  $\lambda$ . This determines, the correct way of taking the sign of the square root: the sign has to be taken so that the right hand side of (5) is everywhere analytic as well, and so that at  $\lambda = 0$  we have  $\mathcal{Z}(0) = 2^n$ , the dimension of the corresponding Fock space. We also see that this means that any zeroes of  $\det(1 + e^{4A})$  must come in pairs, as to not create branch cuts in the complex  $\lambda$  plane. In the next section we will repeat this type of argument with several other expressions involving square roots.

To prove the above results, we need the following preliminary considerations. Consider  $O \in O(2n, \mathbb{C})$ , a complex orthogonal transformation (i.e. such that  $O^T O = 1$  (note: complex orthogonal are very different from unitary, in particular this group is not compact)). We note that the transformation:

$$c_i \rightarrow O_{ij} c_j \quad (6)$$

preserves the anti-commutation relations (2).

Next, we establish the important fact that this transformation can always be written as a *similarity* transformation on the Clifford algebra obeyed by the  $c_i$ , i.e. there is some  $X(O)$ , and operator acting on Fock space, such that  $X(O)c_i X(O)^{-1} = O_{ij} c_j$ . To see this, note that any complex orthogonal transformation can be written as:

$$O = O_R e^{iK} \quad (7)$$

where  $O_R$  is a real orthogonal matrix,  $O_R \in O(2n, \mathbb{R})$  and  $K \in Skew(n, \mathbb{R})$  is a real anti-symmetric matrix.

The transformations  $e^{iK}$  and  $O_R \in SO(2n, \mathbb{R})$  can be generated on the  $c_i$  by applying exponents of bilinears with antisymmetric form using

$$(a) \quad c_m \Rightarrow e^{-\frac{i}{4} K_{ij} c_i c_j} c_m e^{\frac{i}{4} K_{kl} c_k c_l} = (e^{iK})_{ml} c_l \quad (8)$$

for antisymmetric matrices  $K \in Skew(2n, \mathbb{C})$ . Transformations (a) are not enough, as these are restricted to exponents of antisymmetric matrices, which do not cover all complex orthogonal transformations (A simple example for an orthogonal matrix which cannot be written as an exponent of an antisymmetric matrix is  $\sigma_z \in O(2, \mathbb{R})$ ).

To get the full  $O(2n, \mathbb{R})$ , we need to add similarity transformations that change signature. To generate such transformations we note that conjugation by  $c_i$  is a similarity transforma-

tion since  $c_i = c_i^{-1}$ :

$$(b) \quad c_j \Rightarrow c_i c_j c_i = -c_j + 2\delta_{ij} c_i. \quad (9)$$

This transformation multiplies all the Majorana operators by  $-1$  except for  $c_i$  itself. Together (a) and (b) transformations generate all possible  $O \in O(2n, \mathbb{C})$ .

*Proof of Eq (3).* Noting that under the transformation (6), we have  $A_{ij} c_i c_j \rightarrow A_{ij} O_{ik} c_k O_{jl} c_l = (O^T A O)_{kl} c_k c_l$ , we conclude that for any  $O \in O(2n, \mathbb{C})$  the above similarity transformations preserve the trace we can write:

$$\mathcal{Z}(A) = \text{Tr} e^{A_{ij} c_i c_j} = \text{Tr} X e^{A_{ij} c_i c_j} X^{-1} = \text{Tr} e^{X(A_{ij} c_i c_j) X^{-1}} = \mathcal{Z}(O A O^T). \quad (10)$$

For A real antisymmetric, one may proceed easily, by transforming A into a canonical form using the standard orthogonal transformation. It is our interest to show that the form (3) holds also for arbitrary complex antisymmetric matrices, in order to be able to account for complex phases as may appear in mean field superconducting Hamiltonians as well as products of exponents of bilinears.

To establish this, we consider a Gantmacher type decomposition, described in the classic book [41]. This decomposition supplies a complex orthogonal diagonalization of the matrix which is rather complicated: Let K be a rank r anti-symmetric matrix, with elementary divisors  $\epsilon_i$  and corresponding ranks  $f_i$ , then there exists a matrix  $O \in O(2n, \mathbb{C})$  such that

$$K = O K O^{-1} \quad (11)$$

where:

$$K = \oplus K_{\epsilon_i} \quad (12)$$

the  $K$  matrices are of size  $2f_i$ , and they are of the form :

$$K_{ij} = \frac{1}{2} \{ \sum_{j=1}^f (\delta_{j,j+1} + 2\epsilon_i \delta_{j,2f-j+1} - \delta_{f+j,f+j+1} + i(\delta_{j,2f-j} + \delta_{j+1,2f-j+1})) \} - (Transpose) \quad (13)$$

This form is complicated to handle directly, so, as an alternative, we will proceed by first assuming that characteristic numbers are non degenerate. In this case we have  $f = 1$  for all  $\lambda$  and the Gantmacher form reduces to:

$$K = \oplus \begin{pmatrix} 0 & \epsilon_i \\ -\epsilon_i & 0 \end{pmatrix} \quad (14)$$

We note that for *real* symmetric matrices, this type of form is always available (regardless of degeneracy of the divisors), and is the one often used when interested in the partition function of a pairing state. We can now write:

$$\mathcal{Z}(A) = \text{Tre}^{\sum \epsilon_i (c_{2i} c_{2i+1} - c_{2i+1} c_{2i})} = \prod_i 2 \cos 2\epsilon_i \quad (15)$$

where we used that:

$$\text{Tre}^{\epsilon (c_{2i} c_{2i+1} - c_{2i+1} c_{2i})} = 2 \cos 2\epsilon \quad (16)$$

note that:

$$\text{Pf}(e^{K_i} - e^{-K_i}) = \text{Pf}(2 \sin(\epsilon_i) i \sigma_y) = 2 \sin(\epsilon_i) \quad (17)$$

so that

$$(2 \cos 2\epsilon_i) = \frac{\sin(4\epsilon_i)}{\sin(2\epsilon_i)} = \frac{\text{Pf}(e^{4K_i} - e^{-4K_i})}{\text{Pf}(e^{2K_i} - e^{-2K_i})} \quad (18)$$

combining these results together, and using that both Pfaffians and determinants of block matrices are products, we have (3). Alternatively, we can write

$$(2 \cos 2\epsilon_i)^2 = (1 + e^{4i\epsilon_i})(1 + e^{-4i\epsilon_i}) = \det(1 + e^{4K_i})$$

giving us eq. (5).

To complete the argument, we now remove the condition on non-degeneracy of the eigenvalues of the matrix. We claim the formula still holds. Indeed, it follows from the definition of  $\mathcal{Z}(A)$ , that  $\mathcal{Z}(A)$  is manifestly an entire function of the elements  $A_{ij}$ . On the other hand the Pfaffian expression in (3) is also entire. To see this note that as a ratio of holomorphic functions, the only potentially problematic points are points where the denominator goes to zero faster than the numerator, making the ratio singular. However, since we have (using  $\text{Pf}(M)^2 = \det(M)$ ) the relation:

$$\left[ \frac{\text{Pf}(e^{-4A} - e^{4A})}{\text{Pf}(e^{-2A} - e^{2A})} \right]^2 = \frac{\det(e^{-4A} - e^{4A})}{\det(e^{-2A} - e^{2A})} = \det(e^{-2A} + e^{2A})$$

where the right side is manifestly regular, we conclude that this type of singularity can not happen. Therefore, even if we start with a degenerate anti-symmetric matrix, we can perturb it with an arbitrarily small deformation into a non-degenerate matrix, where the equality has been established. By analyticity, we conclude that the identity (3) must hold in the degenerate case as well.

*Remark 1.* We note that the appearance of square roots is a consequence of the nature of the Majorana representation. The topological reason for this is that the Clifford representation of the Lie algebra of skew-symmetric matrices  $Skew_{2n}$ , even for just one fermion mode (two Majoranas) when exponentiated, corresponds to a double cover. This can be seen in the following simple example: Take  $A = 0$  and  $B_{ij} = 2\pi i(\sigma_y)_{ij}$  if  $i, j \in \{1, 2\}$  and  $B_{ij} = 0$  otherwise. Noting that the eigenvalues of  $B$  are just  $\pm 2\pi i$ , we immediately have:

$$e^A = e^B = \mathbf{I}_{2n} \quad (19)$$

However,

$$e^{\frac{1}{4}A_{ij}c_ic_j} = e^0 = \mathbf{I}_{2n}, \quad (20)$$

where  $\mathbf{I}_{2n}, \mathbf{I}_{2^n}$  are the identity matrices in the  $2n$  dimensional mode space and  $2^n$  dimensional Fock space, respectively. On the other hand:

$$e^{\frac{1}{4}B_{ij}c_ic_j} = e^{\frac{\pi}{2}(c_1c_2 - c_2c_1)} \neq \mathbf{I}_{2n} \quad (21)$$

moreover, take  $n = 1$ , then we have:

$$\text{Tr} e^{\frac{\pi}{2}(c_1c_2 - c_2c_1)} = -2 = -\text{Tr} 1 \quad (22)$$

So when writing  $\text{Tr} e^{\frac{1}{4}A_{ij}c_ic_j}$  in terms of properties of  $e^A$ , the information about the sign comes in a subtle way. This is exactly what our Pfaffian formula (3) keeps track of. Indeed, computing the same expression using the Pfaffian we have

$$\text{Tr} e^{\frac{\pi}{2}(c_1c_2 - c_2c_1)} = \frac{\text{Pf}(e^{-2\pi i\sigma_y} - e^{2\pi i\sigma_y})}{\text{Pf}(e^{-\pi i\sigma_y} - e^{\pi i\sigma_y})} = \frac{\text{Pf}(0)}{\text{Pf}(0)} \quad (23)$$

To resolve the ratio, we compute the same expression as a ratio:

$$\lim_{\epsilon \rightarrow 0} \text{Tr} e^{\frac{\pi+\epsilon}{2}(c_1c_2 - c_2c_1)} = \lim_{\epsilon \rightarrow 0} \frac{\text{Pf}(e^{-2(\pi+\epsilon)i\sigma_y} - e^{2(\pi+\epsilon)i\sigma_y})}{\text{Pf}(e^{-(\pi+\epsilon)i\sigma_y} - e^{(\pi+\epsilon)i\sigma_y})} = \lim_{\epsilon \rightarrow 0} \frac{\text{Pf}(-4\epsilon i\sigma_y)}{\text{Pf}(2\epsilon i\sigma_y)} = -2 \quad (24)$$

### III. THE TRACE OF A PRODUCT OF EXPONENTS

For applications it is often more important to understand how to extend the previous results to deal with products of exponents. For this case we have the following trace formula:

$$\text{Tr} e^{A_{1ij}c_ic_j} \dots e^{A_{nij}c_ic_j} = \sqrt{\det(1 + e^{4A_1} \dots e^{4A_n})}. \quad (25)$$

Finally, the expectation value of a product of exponents, in a thermal state with a Hamiltonian  $H_{ij}c_ic_j$  is:

$$\langle e^{A_{1ij}c_ic_j} \dots e^{A_{nij}c_ic_j} \rangle = \sqrt{\det(n_\beta + (1 - n_\beta)e^{4A_1} \dots e^{4A_n})} \quad (26)$$

$$n_\beta = (1 + e^{4\beta H})^{-1} \quad (27)$$

To derivate of Eq (25) we first note that:

$$\left[ \frac{1}{4}K_{lm}c_l c_m, \frac{1}{4}L_{ij}c_i c_j \right] = \frac{1}{4}([K, L])_{im}c_i c_m \quad (28)$$

which is straightforward to check (see, e.g. [42]). Showing that the map:

$$K \rightarrow \frac{1}{4}K_{lm}c_l c_m \quad (29)$$

is a representation of the Lie algebra of skew symmetric matrices. In particular it follows that for small enough  $|t| < t_0$ , we can write:

$$e^{\frac{t}{4}A_{lm}c_l c_m} e^{\frac{t}{4}B_{lm}c_l c_m} = e^{\frac{1}{4}C[tA, tB]_{lm}c_l c_m} \rightarrow e^{tA} e^{tB} = e^{C[tA, tB]} \quad (30)$$

where, for small enough  $C[A, B]$  is given by a Baker Campbell Hausdorff (BCH) type series, we kept  $t$  since in the BCH formula and it's explicit variants such as the Dynkin formula,  $C[A, B]$  as a series in commutators of  $A$  and  $B$ , has in general a finite radius of convergence (see remark bellow). In this neighborhood, we can immediately write:

$$\text{Tr } e^{\frac{t}{4}A_{ij}c_i c_j} e^{\frac{t}{4}B_{ij}c_i c_j} = \text{Tr } e^{\frac{1}{4}C_{ij}c_i c_j} = \sqrt{\det(1 + e^C)} = \sqrt{\det(1 + e^{tA} e^{tB})} \quad (31)$$

we now use analyticity again: the LHS is an entire function of  $t$ , and in particular it's square is. On the other hand  $\det(1 + e^{tA} e^{tB})$  is an entire function as well, since the functions are equal for  $t < t_0$ , they are equal everywhere. It is left to take the square root and resolve the sign so as to be an entire function, which goes to  $2^n$  at  $t = 0$ . Finally, we establish (25) by renaming  $\{A, B\} \rightarrow \{4A, 4B\}$ , and repeating the argument iteratively for an arbitrary number of matrices. Unfortunately, the Pfaffian expression is not available in a simple form anymore, since there is no simple way of expressing the denominator, which requires the square root of  $e^C$ .

*Remark 2.* The question of the range of analyticity of the BCH formula has been studied in many works since the classic paper of Wei [43]. The possible non-analyticity of BCH can be



demonstrated with an example. Following the approach presented in [43], we construct such an example for our particular Lie algebra of complex skew-symmetric matrices by searching for a pair of such matrices where  $[A, B] = A$ . The smallest non-commuting algebra of skew-symmetric matrices is of  $skew_3(\mathbb{C})$ , and we consider the following pair of anti-symmetric matrices:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -i\sqrt{2} \\ -1 & i\sqrt{2} & 0 \end{pmatrix} ; B = \begin{pmatrix} 0 & 1 - i\sqrt{2} & 1 \\ -1 + i\sqrt{2} & 0 & -1 - i\sqrt{2} \\ -1 & 1 + i\sqrt{2} & 0 \end{pmatrix} \quad (32)$$

An explicit computation shows that:

$$e^{a(-2B)}e^{bA} = e^{a(-2B) + \frac{2abe^{2a}}{e^{2a}-1}A} \quad (33)$$

so

$$C[-2aB, bA] = a(-2B) + \frac{2abe^{2a}}{e^{2a}-1}A \quad (34)$$

Clearly, the BCH  $C$  is not analytic in the input matrices  $A, B$  close to  $a = \pi i$ .

To go around this issue, one possible attempt is to use upper triangular matrices  $A, B$  instead of antisymmetric ones, since  $A_{ij}c_ic_j = 2A_{ij}^{up}c_ic_j$  where  $A^{up}$  is the upper triangular part of  $A$ .

The algebra of upper triangular matrices is nilpotent, and as such all BCH type series terminate, and converge. However, it is straightforward to check that the map  $A^{up} \rightarrow A_{ij}^{up}c_ic_j$  is not a representation of the Lie algebra of upper triangular matrices. We do note that upper triangular matrices  $A^{up}$ , can be very useful in obtaining various relations between fermionic traces and determinants or pfaffians. An example for such is Lieb's theorem on pfaffians [44]. An alternative derivation of (3) in a manner similar to Lieb's derivation may be possible.

#### IV. APPLICATIONS TO FULL COUNTING STATISTICS

Here we consider the counting statistics, in systems evolving with a time dependent mean field BCS type hamiltonian as represented by Bogolubov-de Gennes equations. For simplicity, we assume the measured charge at time  $t = 0$  is a good quantum number. More precisely:

Consider measuring an observable  $Q_0$  at time  $t = 0$ , and measuring an observable  $Q_1$  at time  $t = 1$ . Here we take  $Q_1 = Q_0$ , but the extension to  $Q_1 \neq Q_2$  is straightforward. The usual two-measurement protocol for counting statistics of the difference between the measurement of  $Q$  at time  $t = 0$  and time  $t$  is conveniently described by a cumulant generating function of the form:

$$\chi(\lambda) = \sum_{a,b} p(a \rightarrow b; t) e^{-\lambda(Q(a)-Q(b))} = \sum_a p_a \langle a | U^\dagger e^{\lambda \hat{Q}} U e^{-\lambda \hat{Q}} | a \rangle \quad (35)$$

where we assumed that  $|a\rangle$  is a complete sets of eigenstates of  $\hat{Q}$ , and  $p_a$  is the probability  $\langle a | \rho | a \rangle$  to be in state  $|a\rangle$  at  $t = 0$ . In the simplest case, the measured states  $|a\rangle$  are also eigenstates of a system with a quadratic Hamiltonian. Therefore, we will take the initial part of the system measured by  $\hat{Q}$  as non paired. For example, we can think of states of a normal lead which are connected at time  $t = 0$  with a superconductor, or where pairing is turned on at time  $t = 0$ . In this case we have:

$$\chi(\lambda) = \langle U^\dagger e^{\lambda \hat{Q}} U e^{-\lambda \hat{Q}} \rangle_\beta \quad (36)$$

To connect with the formalism above, consider fermions in a Fock space built from the single particle Hilbert space  $\mathfrak{h}$ . We first express bilinear fermion forms in terms of majoranas and using (25). Indeed, consider a general bilinear fermion operator as:

$$\mathcal{M} = \sum_{ij} M_{ij} a_i^\dagger a_j + \frac{1}{2} \left\{ \Delta_{-ij} a_i a_j + \Delta_{+ij} a_i^\dagger a_j^\dagger \right\} \quad (37)$$

where the matrices  $\Delta_-, \Delta_+$  are assumed to be *anti-symmetric* (a choice that can always be made). Note that, in general, we do not demand that the operator  $\mathcal{M}$  is hermitian, hence we do not assume, a priory, a conjugacy relation between  $\Delta_-$  and  $\Delta_+$ , nor do we assume  $M$  to be Hermitian. We rewrite the operator in terms of majoranas as:

$$\mathcal{M} = \frac{1}{4} \mathbb{M}_{ij} c_i c_j + \frac{1}{2} \text{Tr} M \quad (38)$$

with

$$\mathbb{M} = \begin{pmatrix} M_a + \frac{1}{2} (\Delta_- + \Delta_+) & iM_s + \frac{i}{2} (\Delta_- - \Delta_+) \\ -iM_s + \frac{i}{2} (\Delta_- - \Delta_+) & M_a - \frac{1}{2} (\Delta_- + \Delta_+) \end{pmatrix}$$

where  $M_{s/a} = \frac{1}{2}(M \pm M^T)$  are the symmetric and antisymmetric parts of  $M$  and  $\mathbb{M}$  acts on the space  $\mathfrak{h} \otimes \mathbb{C}^2$ . We then have:

$$\text{Tr } \Pi_m e^{\alpha_m \mathcal{M}_m} = e^{\frac{1}{2} \sum_m \text{Tr } M_m} \sqrt{\det (1 + \Pi_m e^{\alpha_m \mathbb{M}_m})}. \quad (39)$$

when the evolution is governed by a Hamiltonian  $\mathcal{H}$ , we can write:

$$\begin{aligned} \chi(\lambda) &= \langle e^{+it\mathcal{H}} e^{\lambda \hat{Q}} e^{-it\mathcal{H}} e^{-\lambda \hat{Q}} \rangle_\beta = \\ &= \det^{1/2} (n_\beta + (1 - n_\beta) e^{it\mathbb{H}} e^{\lambda \mathbb{Q}} e^{-it\mathbb{H}} e^{-\lambda \mathbb{Q}}) \end{aligned} \quad (40)$$

with:

$$n_\beta = \frac{1}{1 + e^{\beta \mathbb{H}_0}} \quad (41)$$

playing the role of the Fermi function of the initial Hamiltonian  $\mathcal{H}_0$ . For a Hermitian operator  $Q$ , we see that for real  $\lambda$ , we must also have  $\chi(\lambda) > 0$ . Therefore, in taking the square root, we take the branch of  $\chi(\lambda)$  which is real and positive everywhere on the real  $\lambda$  axis.

*Remark 3.* In the case where the initial state is not an eigenstate of the charge operator considered, one has to first “decohere” the initial density matrix in the charge basis, which can be implemented by an auxiliary integral).

## V. FULL COUNTING STATISTICS OF THE NUMBER OF FERMIONS IN A REGION

As an example, let us consider the counting statistics of the number of particles in a region  $A$  of a thermal state defined by a BCS type Hamiltonian:

$$\mathcal{H} = \sum_{ij} H_{ij} a_i^\dagger a_j + \frac{1}{2} \Delta_{ij} a_i a_j + h.c. \quad (42)$$

Associated with  $\mathcal{H}$  is a matrix  $\mathbb{H}_0$  given by:

$$\mathbb{H}_0 = \begin{pmatrix} i\text{Im}(\Delta + H) & i\text{Re}(\Delta + H) \\ -i\text{Re}(\Delta - H) & i\text{Im}(H - \Delta) \end{pmatrix} \quad (43)$$

Noting that the number operator in region  $A$  can be written as

$$\hat{N}_A = \sum_{x \in A} a_x^\dagger a_x = \sum_x P_{Ax, x'} a_x^\dagger a_{x'} \quad (44)$$

where  $P_{Ax,x'} = \delta_{xx'}\Theta(x \in A)$  are the matrix elements of  $P_A$ , the single particle projector on region  $A$ . We immediately have, noting that:

$$a_i^\dagger a_i = \quad (45)$$

$$\chi_A(\lambda) \equiv \frac{\text{Tre}^{-\beta\mathcal{H}_0} e^{\lambda\hat{N}_A}}{\mathcal{Z}} = e^{\frac{1}{2}\lambda\text{Tr}P_A} \sqrt{\det \left( \frac{1}{1+e^{\beta\mathbb{H}_0}} + \left(1 - \frac{1}{1+e^{\beta\mathbb{H}_0}}\right) e^{-\lambda P_A \otimes \sigma_y} \right)} \quad (46)$$

where  $\sigma_y$  acts on the auxiliary  $\mathbb{C}^2$  space, so

$$P_A \otimes \sigma_y = \begin{pmatrix} 0 & iP_A \\ -iP_A & 0 \end{pmatrix}.$$

We note that here the sign of the square root of the determinant is unambiguous: as before, it is assured to be positive for real  $\lambda$ , since by definition  $\chi_A(\lambda)$  as defined here is positive.

As an example, for simplicity, we consider a BCS type Hamiltonian (43), and we simply take  $A$  to be the entire volume of the system. We will take  $H = \sum_p h(p) a_p^\dagger a_p$ , where  $h(p)$  may, for example, be taken the form  $h(p) = \frac{p^2}{2m} - \mu$ . Also, take a simple  $s$ -wave  $\Delta$ , only pairing states with momenta  $p$  and  $-p$ . In this case we have to consider each pair of  $p, -p$  separately. For each paired couple with a given  $p$ , we must consider a  $4 \times 4$  block from  $\mathbb{H}_0$  of the form:

$$\begin{pmatrix} 0 & 0 & h(p) & -\Delta(p) \\ 0 & 0 & \Delta(p) & h(p) \\ -h(p) & -\Delta(p) & 0 & 0 \\ \Delta(p) & -h(p) & 0 & 0 \end{pmatrix} \quad (47)$$

Plugging the matrix (47) into (46), and doing some algebra we find:

$$\chi(\lambda, p) = \frac{e^\lambda}{2} \text{sech}^2 \left( \frac{\beta}{2} \sqrt{h^2 + \Delta^2} \right) \times \left( 1 + \cosh \left( \beta \sqrt{h^2 + \Delta^2} \right) \cosh(\lambda) - \frac{h \sinh \left( \beta \sqrt{h^2 + \Delta^2} \right) \sinh(\lambda)}{\sqrt{h^2 + \Delta^2}} \right) \quad (48)$$

and finally:

$$\begin{aligned} \log(\chi(\lambda)) = \int g(p) dp \Big\{ & \lambda + \log \left( 1 + \right. \\ & \cosh \left( \beta \sqrt{h^2 + \Delta^2} \right) \cosh(\lambda) - \frac{h \sinh \left( \beta \sqrt{h^2 + \Delta^2} \right) \sinh(\lambda)}{\sqrt{h^2 + \Delta^2}} \Big) - \\ & \left. \log \left( 2 \cosh^2 \left( \frac{1}{2} \beta \sqrt{h^2 + \Delta^2} \right) \right) \right\} \quad (49) \end{aligned}$$

where  $g(p)$  is the density of pairs at a given  $p$ .

To verify our result, we can also derive this formula directly. Indeed, write the pairing Hamiltonian (suppressing the momentum index  $p$ ),

$$\mathcal{H} = h(a^\dagger a + b^\dagger b) + \Delta (a^\dagger b^\dagger + ba) . \quad (50)$$

Affecting a Bogolubov transformation we can write  $\mathcal{H}$  in the form

$$\mathcal{H} = \mathcal{E} c^\dagger c - \mathcal{E} d^\dagger d \quad ; \quad \mathcal{E} = \sqrt{h^2 + \Delta^2}, \quad (51)$$

with

$$c = ua + vb^\dagger \quad ; \quad d = va - ub^\dagger \quad (52)$$

where

$$u = \cos\left(\frac{1}{2} \tan^{-1}\left(\frac{\Delta}{h}\right)\right) \quad ; \quad v = \sin\left(\frac{1}{2} \tan^{-1}\left(\frac{\Delta}{h}\right)\right) \quad (53)$$

Taking the number operator of the pair  $\hat{N}_A = a^\dagger a + b^\dagger b$ , we have

$$\begin{aligned} \langle e^{\lambda \hat{N}} \rangle &= \langle (1 + (z - 1)a^\dagger a) (1 + (z - 1)b^\dagger b) \rangle = \\ &= (1 - \langle a^\dagger a \rangle - \langle b^\dagger b \rangle + \langle a^\dagger a b^\dagger b \rangle) + \\ &+ z (\langle a^\dagger a \rangle + \langle b^\dagger b \rangle - 2 \langle a^\dagger a b^\dagger b \rangle) + z^2 \langle a^\dagger a b^\dagger b \rangle \end{aligned} \quad (54)$$

where we denoted  $z = e^\lambda$ . Substituting  $a = uc - vd$  and  $b^\dagger = vc + ud$  and computing the resulting thermal correlations, which are just free fermions in terms of the  $c$  and  $d$  operators, we find:

$$\begin{aligned} \chi &= (v^2 n_c + u^2 n_d - n_c n_d) + z (1 - n_d - n_c + 2 n_d n_c) + \\ &+ z^2 (v^2 n_d + u^2 n_c - n_d n_c) \end{aligned} \quad (55)$$

with  $n_d, n_c$  are the fermi functions for  $c, d$ , i.e.:

$$n_d = \frac{1}{1 + e^{-\beta \mathcal{E}}} \quad ; \quad n_c = \frac{1}{1 + e^{\beta \mathcal{E}}}. \quad (56)$$

After substituting  $z$  and  $\mathcal{E}$ , and some tedious algebra we recover the result (48).

## VI. OVERLAPS OF PAIRED STATES AND COUNTING STATISTICS OF CHARGE

Here we show how to compute the overlap between different BCS like states built out of the vacuum. Such a state is built out of application of pair creation to the vacuum. We will write a (un-normalized) state of this type in the form:

$$|D\rangle = e^{D_{ij}a_i^\dagger a_j^\dagger}|0\rangle, \quad (57)$$

where the state  $|0\rangle$  is the vacuum state, so that  $a_i|0\rangle = 0$  for all  $i$ , and  $D$  is an  $n \times n$  antisymmetric matrix. We now derive the following formulas:

$$\langle D'|D\rangle = (-1)^n Pf \begin{pmatrix} D'^\dagger & -I_n \\ I_n & D \end{pmatrix} \quad (58)$$

and as a corollary:

$$\langle e^{i\lambda N_A}\rangle_D = \frac{(-1)^n}{\det(1 + D^\dagger D)} Pf \begin{pmatrix} D^\dagger & -I_n \\ I_n & e^{i\lambda P_A} D e^{i\lambda P_A} \end{pmatrix} \quad (59)$$

To derive these relations, we first note that  $|0\rangle$  is the ground state of the Hamiltonian  $H = \hat{N} = \sum_i a_i^\dagger a_i$ . We can therefore write the overlap in the following way:

$$\langle D'|D\rangle = \lim_{\beta \rightarrow \infty} \frac{1}{Z} Tr[e^{D_{ij}a_i^\dagger a_j^\dagger} e^{-\beta \hat{N}} (e^{D'_{ij}a_i^\dagger a_j^\dagger})^\dagger] \quad (60)$$

Going to the Majorana representation we have:

$$D_{ij}a_i^\dagger a_j^\dagger = \frac{1}{8}[D \otimes (\sigma_z - i\sigma_x)]_{\alpha\beta} c_\alpha c_\beta \quad (61)$$

$$D_{ij}a_i a_j = \frac{1}{8}[D \otimes (\sigma_z + i\sigma_x)]_{\alpha\beta} c_\alpha c_\beta \quad (62)$$

$$\sum_i a_i^\dagger a_i = -\frac{1}{4}I_n \otimes \sigma_y + \frac{1}{2}Tr I_n. \quad (63)$$

We have seen above (25), that:

$$\frac{1}{Z} Tr e^{-\beta N} \left( e^{D'_{ij}a_i^\dagger a_j^\dagger} \right)^\dagger e^{D_{ij}a_i^\dagger a_j^\dagger} = \sqrt{\det \left( 1 - n_\beta + n_\beta e^{\frac{1}{2}(D'^\dagger \otimes (\sigma_z + i\sigma_x))} e^{\frac{1}{2}(D \otimes (\sigma_z - i\sigma_x))} \right)}. \quad (64)$$

To proceed we make several observations.

1. notice that  $(\sigma_z \pm i\sigma_x)^2 = 0$  are nilpotent, allowing us to write:

$$e^{\frac{1}{2}(D'^\dagger \otimes (\sigma_z + i\sigma_x))} e^{\frac{1}{2}(D \otimes (\sigma_z - i\sigma_x))} = \left( 1 + \frac{1}{2} (D'^\dagger \otimes (\sigma_z + i\sigma_x)) \right) \left( 1 + \frac{1}{2} (D \otimes (\sigma_z - i\sigma_x)) \right) \quad (65)$$

2. That:

$$\lim_{\beta \rightarrow \infty} n_\beta = \lim_{\beta \rightarrow \infty} \frac{1}{1 + e^{4\beta(-\frac{1}{4}I_n \otimes \sigma_y)}} = \frac{1}{2} (I_{2n} + I_n \otimes \sigma_y) = \frac{1}{2} I_n \otimes (1 + \sigma_y) \equiv P_+ \quad (66)$$

so that

$$\langle D|D' \rangle = \sqrt{\det \left( P_+ + P_+ \left( 1 + \frac{1}{2} (D'^\dagger \otimes (\sigma_z + i\sigma_x)) \right) \left( 1 + \frac{1}{2} (D \otimes (\sigma_z - i\sigma_x)) \right) P_+ \right)} \quad (67)$$

3. Next, we note that:

$$P_+ (\sigma_z \pm i\sigma_x) P_+ = 0 \quad (68)$$

and

$$P_+ (\sigma_z + i\sigma_x) (\sigma_z - i\sigma_x) P_+ = 2I_n \otimes (1 + \sigma_y) = 4P_+ \quad (69)$$

combining these we have:

$$\langle D|D' \rangle = \sqrt{\det (P_+ (1 + D'^\dagger D \otimes I_2) P_+)} = \sqrt{\det_n (1 + D'^\dagger D)}. \quad (70)$$

Using the rules of determinants of block matrices we can also rewrite the last expression as:

$$\langle D|D' \rangle = \sqrt{\det \begin{pmatrix} D'^\dagger & -I \\ I & D \end{pmatrix}} \quad (71)$$

We now finally have a determinant of an anti-symmetric matrix, and the identity  $Pf(A)^2 = \det(A)$  (valid for anti-symmetric matrices, but not true for general matrices) can be safely used, to get:

$$\langle D|D' \rangle = (-1)^n Pf \begin{pmatrix} D'^\dagger & -I \\ I & D \end{pmatrix} \quad (72)$$

where the phase  $(-1)^n$  was added by demanding  $\langle 0|0 \rangle = 1$  and using:

$$Pf \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = (-1)^n. \quad (73)$$

This establishes (58).

Finally, to get equation (59), we write:

$$\langle e^{i\lambda N_A} \rangle_D = \frac{1}{\langle D|D \rangle} \langle D|e^{i\lambda N_A}|D \rangle = \frac{1}{\langle D|D \rangle} \langle D|e^{i\lambda P_A} D e^{i\lambda P_A} \rangle \quad (74)$$

where the last equation is a consequence of

$$e^{i\lambda N_A} a_i^\dagger e^{-i\lambda N_A} = \begin{cases} e^{i\lambda} a_i^\dagger & i \in A \\ a_i^\dagger & i \notin A \end{cases}. \quad (75)$$

Now using (58) in gives (59).

## VII. PARITY FLUCTUATIONS

Consider the distribution of the parity operator applied to a subset  $A$  of modes of the (complex-fermions)  $a_i$ . In systems with pairing, such parity measurements may be associated with topological effects, and are effected by the presence of majorana zero modes, see, e.g. [45]. The parity operator can be written as:

$$\mathcal{P} = (-1)^{\hat{N}_A} = e^{\pi i \hat{N}_A} . \quad (76)$$

Noting that  $\langle \mathcal{P} \rangle = p_e - p_o = 2p_e - 1$ , where  $p_e, p_o$  is the probability that the parity of the fermion number is even/odd. It's variance is given by:

$$\sigma^2 = \langle \mathcal{P}^2 \rangle - \langle \mathcal{P} \rangle^2 = 1 - \langle \mathcal{P} \rangle^2 \quad (77)$$

We immediately see that:

$$\sigma^2 = 1 - \langle \mathcal{P} \rangle^2 = 1 - \det (1 - n_\beta + n_\beta e^{-i\pi\sigma_y P_A}) \quad (78)$$

This expression can be further simplified using:

$$e^{-i\pi\sigma_y P_A} = \mathbf{I}_{2n} - 2P_A \otimes \mathbf{I}_2 \quad (79)$$

to get

$$\sigma^2 = 1 - \det (1 - 2n_\beta P_A \otimes \mathbf{I}_2) \quad (80)$$

The expectation value of the parity itself can be computed for states such as  $|D\rangle$  in (59) by simply plugging in  $\lambda = \pi$ . One can also derive pfaffian forms for the case not covered in (59), however here we will proceed in the simplest way using the determinant formulas above.

We can write it down in the following way. If  $\hat{N}_A = \sum_{m \in A} \hat{N}_m$  for some set of modes, with  $\hat{N}_m$  is the number operator associated with mode  $m$ , then we use the relation

$$e^{i\pi\hat{N}_m} = 1 - 2\hat{N}_m = (1 - 2\partial_\lambda) e^{\lambda\hat{N}_m} |_{\lambda=0} \quad (81)$$

to write

$$\langle \mathcal{P} \rangle = \frac{1}{Z} \text{Tr} e^{-\beta H_{ij} c_i c_j} \prod_{m \in A} e^{i\pi\hat{N}_m} = \frac{1}{Z} \prod_{m \in A} (1 - 2\frac{\partial}{\partial \lambda_i}) \text{Tr} e^{-\beta H_{ij} c_i c_j} \prod_{k \in A} e^{\lambda_k \hat{N}_k} |_{\{\lambda\}=0} \quad (82)$$



We can finally write this expression as:

$$\langle \mathcal{P} \rangle = \prod_{m \in A} (1 - 2 \frac{\partial}{\partial \lambda_m}) e^{\frac{1}{2} \sum_{i \in A} \lambda_i \det^{1/2} (n_\beta + (1 - n_\beta) e^{\sum_{i \in A} \lambda_i \sigma_y P_i})} |_{\{\lambda\}=0} \quad (83)$$

there is no sign ambiguity in this last expression since for Hermitian  $H$  and real  $\lambda$ ,  $\text{Tr} e^{-\beta H_{ij} c_i c_j} \prod_{m \in A} e^{\lambda_i \hat{N}_m}$  should always be positive. In topological applications one considers Majorana zero modes as the simplest known example of non-abelian particles,  $P_A$  can be taken to be the rank one projection operator on the Dirac fermion mode consisting of the two unpaired Majoranas, and thus their “state” is determined by the parity. For this type of applications the formula above works very well, since we deal with a particular mode or two, and one can analytically carry out the derivatives above.

### VIII. FINAL REMARKS

In this paper we have summarized a few formulas of traces for exponentials of fermion bi-liners which include pairing terms, and are not necessarily hermitian. We are not aware of previous appearance in the literature of the “sign ambiguity free” formulas (3) and (59).

We believe that the (perhaps more practically useful) expressions such as (25) and (26) may have appeared in various forms in dealing with concrete problems, however we feel it is useful to give them a general framework and a simple proof, and make them available for other types of problems. Indeed, the above expressions can be straightforwardly applied to numerical and analytical investigations of time dependent problems involving fermions, such as the extension of the study of quasi-particle modeling of X-ray absorption in the cuprates [46, 47] to take into account the presence of pairing terms [48]. The formulas

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